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Korovkin-Type Approximation on C^*-Algebras

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THEOREM: Let A and B be complex C^*-algebras with identities 1_A and 1_B, respectively. Let (ϕ_n) be a sequence of positive linear maps with $\phi_n(1_A) \leqslant 1_B$ from A to B, and ϕ a C^*-homomorphism from A to B. Then $C = \{a \in A: \phi_n(a) \to \phi(a), \phi_n(a^* \circ a) \to \phi(a^* \circ a)\}$ is a norm-closed *subspace of A and is closed under the Jordan product $a \circ b = (ab + ba)/2$ in A. If all ϕ_n and ϕ are Schwarz maps, then C is, in fact, a C^*-subalgebra of A. Here $\to$ denotes the operator norm convergence, or the weak operator convergence, or the strong operator convergence. A modification of this theorem for convergence in the trace norm is also considered. Various examples are given to illustrate the theorem.

For a complex C^*-algebra A with identity 1_A and a sequence (ϕ_n) of positive linear maps from A to A with $\phi_n(1_A) \leqslant 1_A$, Priestley proved in [5] that the set

$$\{a \in A: \phi_n(a) \to a, \phi_n(a^2) \to a^2, \phi_n(a^* \circ a) \to a^* \circ a\}$$

is a J^*-subalgebra of A (i.e., a norm-closed *subspace of A which is closed under the Jordan product $a \circ b = (ab + ba)/2$), where $\to$ denotes convergence in the operator norm topology, or in the weak or strong operator topologies. Later, Robertson proved in [6] that if each ϕ_n is a Schwarz map (i.e., $\phi_n(a)^* \phi_n(a) \leqslant \phi_n(a^*a)$ for all $a \in A$), then the set

$$\{a \in A: \phi_n(a) \to a, \phi_n(a^*a) \to a^*a, \phi_n(aa^*) \to aa^*\}$$

is, in fact, a C^*-subalgebra of A, where $\to$ denotes the operator norm convergence.

The purpose of the present paper is to improve (i) Priestley's result by dropping the condition $\phi_n(a^2) \to a^2$ in it, and also (ii) Robertson's result by replacing the conditions $\phi_n(a^*a) \to a^*a$, $\phi_n(aa^*) \to aa^*$ in it by the condition $\phi_n(a^* \circ a) \to a^* \circ a$. Our results are deduced from a preliminary theorem of Priestley which we shall quote later.

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Also, we shall work in a more general situation. Throughout this note, A and B will denote complex C^*-algebras with identities 1_A and 1_B, respectively. A *linear map $\phi: A \to B$ is called a C^*-homomorphism if $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for all a, b in A. C^*-homomorphisms are important because they preserve the quantum mechanical properties of the C^*-algebras. We consider an approximation of a C^*-homomorphism $\phi: A \to B$ by a sequence of positive linear maps $\phi_n: A \to B$ satisfying $\phi_n(1_A) \leqslant 1_B$. Unless otherwise stated, $\to$ will denote convergence in the operator norm topology, or in the weak or strong operator topologies.

We begin by proving a convergence result for the C^*-algebra $\beta(H)$ of all bounded operators on a complex Hilbert space H.

LEMMA 1. *Let (S_n), (T_n), (U_n) and (V_n) be sequences in $\beta(H)$ such that for all n,*

$$S_n^* S_n \leqslant U_n \quad \text{and} \quad T_n^* T_n \leqslant V_n.$$

If $S, T \in \beta(H)$ and

$$S_n \to S, \qquad T_n \to T, \qquad U_n + V_n \to S^*S + T^*T,$$

then

$$U_n \to S^*S \quad \text{and} \quad V_n \to T^*T,$$

where $\to$ denotes either the operator norm convergence or the weak operator convergence.

If $\to$ denotes the strong operator convergence, then this result is false in general, but remains true in the following two special cases: (i) *S_n and T_n are self-adjoint for all n,* (ii) *$T_n^* = S_n$ for all n.*

Proof. For all n,

$$(U_n - S_n^* S_n) + (V_n - T_n^* T_n) = (U_n + V_n) - (S^*S + T^*T) - (S_n^* S_n - S^*S) - (T_n^* T_n - T^*T).$$

If we show that $S_n^* S_n \to S^*S$ and $T_n^* T_n \to T^*T$, then the right side of the above equality tends to zero. But for each n the left side is the sum of two positive operators so that $U_n - S_n^* S_n \to 0$ and $V_n - T_n^* T_n \to 0$; i.e., $U_n \to S^*S$ and $V_n \to T^*T$.

If $\to$ denotes the operator norm convergence, then since $S_n \to S$ and $T_n \to T$ it is obvious that $S_n^* S_n \to S^*S$ and $T_n^* T_n \to T^*T$.

Let, now, $\rightarrow$ denote the weak operator convergence. Fix $x \in H$. Then

$$\begin{aligned}\|S_n(x) - S(x)\|^2 + \|T_n(x) - T(x)\|^2 \\ &= \langle (S_n^* S_n + T_n^* T_n)(x), x\rangle + \langle (S^*S + T^*T)(x), x\rangle \\ &\quad - 2\,\mathrm{Re}\langle S(x), S_n(x)\rangle - 2\,\mathrm{Re}\langle T(x), T_n(x)\rangle \\ &\leqslant \langle (U_n + V_n)(x), x\rangle + \langle (S^*S + T^*T)(x), x\rangle \\ &\quad - 2\,\mathrm{Re}\langle S(x), S_n(x)\rangle - 2\,\mathrm{Re}\langle T(x), T_n(x)\rangle,\end{aligned}$$

which tends to zero. Hence $\|S_n(x) - S(x)\| \rightarrow 0$ and $\|T_n(x) - T(x)\| \rightarrow 0$. Now,

$$\langle S_n^* S_n(x), x\rangle = \|S_n(x)\|^2 \rightarrow \|S(x)\|^2 = \langle S^*S(x), x\rangle.$$

Since $x \in H$ is arbitrary, we see that $S_n^* S_n \rightarrow S^*S$, and similarly $T_n^* T_n \rightarrow T^*T$.

If $\rightarrow$ denotes the strong operator convergence, the statement of the lemma is false as the following example shows. Let $H = l^2$, the Hilbert space of square-summable complex sequences, and L denote the unilateral left shift operator on l^2. Define $S_n = I - L^n$, $T_n = I + L^n$, $U_n = S_n^* S_n$ and $V_n = T_n^* T_n$. Then it can be readily seen that $S_n \rightarrow I$, $T_n \rightarrow I$ and $U_n + V_n = 2(I + L^{*n}L^n) \rightarrow 2I$ strongly, but U_n does not tend to I strongly. If, however, all S_n and T_n are self-adjoint, then clearly $S_n^* S_n = S_n^2 \rightarrow S^2 = S^*S$ and similarly $T_n^* T_n \rightarrow T^*T$. Again, if $T_n^* = S_n$ for each n, then $S_n^* S_n = T_n S_n \rightarrow TS = S^*S$ and $T_n^* T_n = S_n T_n \rightarrow ST = T^*T$, and we are back in business. ∎

We now state the preliminary theorem of Priestley from which our results will be deduced. Although he proved it for the special case $A = B$ and ϕ equal to the identity map, his proof works equally well in the general situation.

THEOREM (Priestley [5, Theorems 1.1 and 2.1]). *The set*

$$J = \{a \in A : a^* = a, \phi_n(a) \rightarrow \phi(a), \phi_n(a^2) \rightarrow \phi(a^2)\}$$

is a Jordan subalgebra of self-adjoint elements of A.

THEOREM 2. *The set*

$$C = \{a \in A : \phi_n(a) \rightarrow \phi(a), \phi_n(a^* \circ a) \rightarrow \phi(a^* \circ a)\}$$

is a J^*-subalgebra of A.

If ϕ and all ϕ_n are Schwarz maps, then C is, in fact, a C^*-subalgebra of A.

Proof. To show that C is a J^*-subalgebra of A, it is enough to prove, by Priestley's theorem, that $C = J + iJ$. Since C obviously contains $J + iJ$, we only have to show that if $a \in C$ and $a = a_1 + ia_2$ with $a_1^* = a_1$, $a_2^* = a_2$, then $a_1, a_2 \in J$.

For all n, let $S_n = \phi_n(a_1)$, $T_n = \phi_n(a_2)$, $U_n = \phi_n(a_1^2)$ and $V_n = \phi_n(a_2^2)$. Then, by Kadison's inequality [3],

$$S_n^* S_n = S_n^2 = \phi_n(a_1)^2 \leqslant \phi_n(a_1^2) = U_n,$$

and similarly, $T_n^* T_n \leqslant V_n$. Also, since $a \in C$ and C is *closed, we see that $S_n \to \phi(a_1) = S$, $T_n \to \phi(a_2) = T$, and

$$U_n + V_n = \phi_n(a_1^2 + a_2^2) = \phi_n(a^* \circ a) \to \phi(a^* \circ a) = S^*S + T^*T.$$

Hence, by Lemma 1,

$$U_n = \phi_n(a_1^2) \to S^*S = \phi(a_1)^2 = \phi(a_1^2)$$

and

$$V_n = \phi_n(a_2^2) \to T^*T = \phi(a_2)^2 = \phi(a_2^2).$$

Thus, $a_1, a_2 \in J$ so that $C = J + iJ$ is a J^*-subalgebra of A.

Next, let ϕ and ϕ_n be Schwarz maps. Since ϕ is also a C^*-homomorphism, it follows that (Corollary 1, p. 270 of [4]) ϕ is, in fact, a *homomorphism. To show that C is a C^*-subalgebra of A, it is enough to prove that if $a_1, a_2 \in C$ with $a_1^* = a_1$ and $a_2^* = a_2$, then $a_1 a_2 \in C$. Let $a = a_1 + ia_2$. Since

$$2a_1 a_2 = (2a_1 \circ a_2 + i(a_1^2 + a_2^2 - a^*a))$$

and since C is a J^*-subalgebra, we need only prove that $a^*a \in C$.

For all n, let $S_n = T_n^* = \phi_n(a)$, $U_n = \phi_n(a^*a)$ and $V_n = \phi_n(aa^*)$. Since ϕ_n is Schwarz, $S_n^* S_n \leqslant U_n$ and $T_n^* T_n \leqslant V_n$. Now $a \in C$ since C is a J^*-subalgebra. Thus, $S_n \to \phi(a) = S$, $T_n \to \phi(a^*) = T$ and

$$\begin{aligned} U_n + V_n = \phi_n(a^*a + aa^*) &= 2\phi_n(a^* \circ a) \to 2\phi(a^* \circ a) \\ &= 2\phi(a^*) \circ \phi(a) = S^*S + T^*T. \end{aligned}$$

Hence, by Lemma 1,

$$\phi_n(a^*a) = U_n \to S^*S = \phi(a)^* \phi(a) = \phi(a^*a),$$

and

$$\phi_n(aa^*) = V_n \to T^*T = \phi(a)\,\phi(a)^* = \phi(aa^*).$$

Again, since a belongs to the J^*-subalgebra C, we see that

$$a^*aa^* = 2a^* \circ (a^* \circ a) - (a^* \circ a^*) \circ a$$

as well as $a \circ (a^*aa^*)$ belong to C. Therefore, $\phi_n(a \circ (a^*aa^*)) \to \phi(a \circ (a^*aa^*))$. But

$$2a \circ (a^*aa^*) = (a^*a)^2 + (aa^*)^2.$$

Now, letting $S_n = \phi_n(a^*a)$, $T_n = \phi_n(aa^*)$, $U_n = \phi_n((a^*a)^2)$ and $V_n = \phi_n((aa^*)^2)$ in Lemma 1, and noting that $S_n \to \phi(a^*a) = S$, $T_n \to \phi(aa^*) = T$ and

$$U_n + V_n = 2\phi_n(a \circ (a^*aa^*)) \to 2\phi(a \circ (a^*aa^*)) = S^*S + T^*T$$

we conclude that

$$\phi_n((a^*a)^2) = U_n \to S^*S = \phi(a^*a)^2 = \phi((a^*a)^2).$$

This shows that $a^*a \in C$ and completes the proof. ∎

Remarks 3. (i) The set C is the same for both weak and strong operator convergence. This follows because C is an algebra and because Theorem 2.1 of [5] shows that the self-adjoint elements in C are the same regardless of whether $\to$ denotes weak or strong convergence.

(ii) A variant of Theorem 2 holds for the algebras of trace class operators. Let $\mathscr{S}$ and $\mathscr{T}$ denote the sets of all trace class operators on complex Hilbert spaces H and K, respectively. Then $\mathscr{S}$ and $\mathscr{T}$ are complex Banach algebras under the respective trace norms $\| \ \|_{\mathscr{S}}$ and $\| \ \|_{\mathscr{T}}$. Let $\phi\colon \mathscr{S} \to \mathscr{T}$ be a *linear map which preserves squares, and for $n = 1, 2, \ldots$, let $\phi_n\colon \mathscr{S} \to \mathscr{T}$ be a positive linear map such that for every $S \in \mathscr{S}$ with $\|S\| \leqslant 1$, we have $\|\phi_n(S)\| \leqslant 1$ and for every $S \in \mathscr{S}$ with $\|S\|_{\mathscr{S}} \leqslant 1$ we have $\|\phi_n(S)\|_{\mathscr{T}} \leqslant \alpha$ for some α independent of n and S. Then the set

$$\{S \in \mathscr{S} : \|\phi_n(S) - \phi(S)\|_{\mathscr{T}} \to 0, \|\phi_n(S^* \circ S) \to \phi(S^* \circ S)\|_{\mathscr{T}} \to 0\}$$

is a $\| \cdot \|_{\mathscr{S}}$-closed Jordan *subalgebra of $\mathscr{S}$; if the ϕ_n's are Schwarz maps and ϕ is a *homomorphism, then it is, in fact, a *subalgebra of $\mathscr{S}$. This result can be deduced from the corresponding result of Priestley (Theorem 3.1 of [5]) since our Lemma 1 and the first two paragraphs of its proof hold verbatim if we replace $\beta(H)$ by the set of all trace class operators on H and let $\to$ denote convergence in the trace norm.

Theorem 2 can be used to obtain the convergence of an approximation method (ϕ_n) on a set larger than the one on which the convergence is known a priori. Thus, if the convergence is known to hold on the $2m$ elements

$a_1,\ldots,a_m$, $a_1^* \circ a_1,\ldots,a_m^* \circ a_m$, then it holds on the J^*-subalgebra (or on the C^*-subalgebra, if the ϕ_n's are Schwarz) generated by $a_1,\ldots,a_m$. In practice, it is even enough to assume convergence on $(m+1)$ elements as the following result shows.

COROLLARY 4. *Let $a_1,\ldots,a_m \in A$ such that*

$$\phi_n(a_j) \to \phi(a_j), \qquad j = 1,\ldots, m,$$

and

$$\phi_n\left(\sum_{j=1}^{m} a_j^* \circ a_j\right) \to \phi\left(\sum_{j=1}^{m} a_j^* \circ a_j\right).$$

Then $\phi_n(a) \to \phi(a)$ for all a belonging to the J^*-subalgebra generated by $a_1,\ldots,a_m$. If all ϕ_n and ϕ are Schwarz maps, then $\phi_n(a) \to \phi(a)$ for all a belonging to the C^*-subalgebra generated by $a_1,\ldots,a_m$.

Proof. By Theorem 2, it is enough to show that

$$\phi_n(a_j^* \circ a_j) \to \phi(a_j^* \circ a_j), \qquad j = 1,\ldots, m.$$

For all n, $\phi_n(a_j^* \circ a_j) - \phi_n(a_j)^* \circ \phi_n(a_j) \geqslant 0$ by Kadison's (generalized) inequality, and the proof will therefore be accomplished by demonstrating that the sum $\sum_{j=1}^m (\phi_n(a_j^* \circ a_j) - \phi_n(a_j)^* \circ \phi_n(a_j))$ tends to zero. Note that this sum is equal to

$$\sum_{j=1}^{m} (\phi_n(a_j^* \circ a_j) - \phi(a_j)^* \circ \phi(a_j)) \\ - \sum_{j=1}^{m} (\phi_n(a_j)^* \circ \phi_n(a_j) - \phi(a_j)^* \circ \phi(a_j)).$$

Since $\phi_n(a_j) \to \phi(a_j)$, it follows that $\phi_n(a_j)^* \circ \phi_n(a_j) \to \phi(a_j)^* \circ \phi(a_j)$ for $j = 1,\ldots, m$. This is obvious for the operator norm convergence, and for the weak or strong operator convergence, we argue as follows. Let $\phi_n(a_j) \to \phi(a_j)$ weakly, $j = 1,\ldots, m$. For any x in the underlying Hilbert space,

$$\tfrac{1}{2} \sum_{j=1}^{m} (\|(\phi_n(a_j) - \phi(a_j))(x)\|^2 + \|(\phi_n(a_j) - \phi(a_j))^*(x)\|^2) \\ \leqslant \left\langle \left(\sum_{j=1}^{m} \phi_n(a_j^* \circ a_j)(x), x\right)\right\rangle + \left\langle \left(\sum_{j=1}^{m} \phi(a_j^* \circ a_j)(x), x\right)\right\rangle \\ - \sum_{j=1}^{m} \mathrm{Re}\langle \phi(a_j)(x), \phi_n(a_j)(x)\rangle - \sum_{j=1}^{m} \mathrm{Re}\langle \phi(a_j)^*(x), \phi_n(a_j)^*(x)\rangle,$$

which tends to zero. Hence $\phi_n(a_j) \to \phi(a_j)$ and $\phi_n(a_j)^* \to \phi(a_j)^*$ strongly. Thus, $\phi_n(a_j)^* \circ \phi_n(a_j) \to \phi(a_j)^* \circ \phi(a_j)$ strongly and the desired result follows. ▮

EXAMPLES 5. (i) Let $A = B = M_k$, the C^*-algebra of all $k \times k$ matrices with complex entries, and let ϕ be the identity map. Let

$$a = \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & \ddots & \\ & 0 & 0 & \\ & & & 0 \end{bmatrix},$$

where the 0's occur on and below the main diagonal and the 1's occur above it. Then

$$2a^* \circ a = \begin{bmatrix} k-1 & k-2 & \cdots & 1 & 0 \\ k-2 & k-1 & \cdots & 2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \cdots & k-1 & k-2 \\ 0 & 1 & \cdots & k-2 & k-1 \end{bmatrix} = [a_{i,j}],$$

where we note that $a_{i,j} = a_{k+1-j,k+1-i}$. It can be seen that the J^*-subalgebra generated by a in M_k is

$$J = \{[b_{i,j}]: b_{i,j} = b_{k+1-j,k+1-i}\},$$

while the C^*-subalgebra generated by a is all of M_k. Hence, by Theorem 2, if for $n = 1, 2,..., \phi_n: M_k \to M_k$ is a positive linear map with $\phi_n(I) \leqslant I$ (resp., a Schwarz map), and $\phi_n(a) \to a$, $\phi_n(a^* \circ a) \to a^* \circ a$, then $\phi_n(b) \to b$ for all $b \in J$ (resp., for all $b \in M_k$).

(ii) Let X be a compact Hausdorff topological space, and $A = B = C(X)$, the C^*-algebra of all continuous complex-valued functions on X with the supremum norm. For $n = 1, 2,...,$ let $\phi_n: C(X) \to C(X)$ be a positive linear map and let $\phi_n(1) \to 1$ uniformly on X. Let $f_1,..., f_m \in C(X)$ separate the points of X and satisfy

$$\phi_n(f_j) \to f_j,$$
$$\phi_n(|f_1|^2 + \cdots + |f_m|^2) \to |f_1|^2 + \cdots + |f_m|^2$$

uniformly on X. Then $\phi_n(f) \to f$ uniformly on X for all $f \in C(X)$.

This can be seen as follows. By considering $\phi'_n = \phi_n / \|\phi_n(1)\|$, we can assume without loss of generality that $\phi_n(1) \leqslant 1$ for all n. For $f \in C(X)$ and $x \in X$,

$$|\phi_n(f)(x)|^2 \leqslant \phi_n(1)(x)\, \phi_n(|f|^2(x)) \leqslant \phi_n(|f|^2)(x),$$

by the classical Cauchy–Schwarz inequality. Hence each ϕ_n is a Schwarz map. By Corollary 4, $\phi_n(f) \to f$ uniformly for all f in the uniformly closed conjugate-closed subalgebra generated by $\{1, f_1, ..., f_m\}$. But it equals $C(X)$, by the Stone–Weierstrass theorem. Hence the result.

In particular, if X is a bounded closed subset of the Euclidean space R^m, then we can consider the coordinate functions

$$f_j(x_1, ..., x_m) = x_j, \qquad j = 1, ..., m.$$

Or, if X is a closed subset of the m-dimensional torus $T^m = \{t = (e^{i\theta_1}, ..., e^{i\theta_m}): 0 \leqslant \theta_j < 2\pi\}$, then we can consider

$$f_j(t) = \cos\theta_j, \; g_j(t) = \sin\theta_j, \qquad j = 1, ..., m.$$

Note here that $|f_1|^2 + \cdots + |f_m|^2 + |g_1|^2 + \cdots + |g_m|^2 = m$.

(iii) Let H be a complex Hilbert space and $A = B = \beta(H)$. Consider a *homomorphism $\phi: \beta(H) \to \beta(H)$. (For example, if $U \in \beta(H)$ is an isometry, then $\phi(T) = UTU^*$ for $T \in \beta(H)$, is a *homomorphism.) Let $\phi_n: \beta(H) \to \beta(H)$ be Schwarz maps and let $S_1, ..., S_m \in \beta(H)$ satisfy

$$\phi_n(S_j) \to \phi(S_j), \qquad j = 1, ..., m,$$

and

$$\phi_n(S_1^* \circ S_1 + \cdots + S_m^* \circ S_m) \to \phi(S_1^* \circ S_1 + \cdots + S_m^* \circ S_m).$$

Assume that $S_1, S_1^*, ..., S_m, S_m^*$ do not have a common non-trivial invariant subspace of H and that the C^*-subalgebra C generated by $S_1, ..., S_m$ contains a non-zero compact operator. Then

$$\phi_n(S) \to \phi(S)$$

for all compact operators S in $\beta(H)$. This follows from Corollary 4 since being an irreducible C^*-subalgebra of $\beta(H)$, C contains all compact operators (Corollary 2, p. 18 of [1]).

To give specific, concrete cases, let $H = L^2([0, 1])$. First consider the *Volterra integration operator*

$$V(f)(x) = \int_0^x f(t)\, dt, \qquad f \in L^2([0, 1]).$$

Then the invariant subspaces of V are of the form $\{f \in L^2([0,1]): f=0$ almost everywhere on $[0,\alpha]\}$, where $0 \leqslant \alpha \leqslant 1$. (See pp. 94–97 of [2].) Hence V and V^* have no non-trivial common invariant subspace. Also, V is itself a compact operator. Hence we can take $m=1$ and $S_1 = V$ in the above result. Thus, if convergence holds on V and $V^* \circ V$, where

$$V^* \circ V(f)(x) = \tfrac{1}{2}\left[\int_0^x \left(\int_y^1 f(t)\,dt\right) dy + \int_x^1 \left(\int_0^y f(t)\,dt\right) dy\right],$$

then it holds on all compact operators on $L^2([0,1])$.

Next, consider the *multiplication* operator

$$M(f)(x) = xf(x), \qquad f \in L^2([0,1]),$$

and if $g(x) = x$ for all $x \in [0,1]$, let

$$T(f) = \left(\int_0^1 f(t)\,dt\right) g, \qquad f \in L^2([0,1]).$$

Then M and M^* have common non-trivial invariant subspaces since $M^* = M$. Also, since

$$T^*(f)(x) = \int_0^1 tf(t)\,dt, \qquad f \in L^2([0,1])$$

it follows that T and T^* leave the subspace $\{f \in L^2([0,1]): \int_0^1 f(t)\,dt = 0 = \int_0^1 tf(t)\,dt\}$ invariant. We show that M and T do not have a common non-trivial invariant subspace. Let F be a closed subspace of $L^2([0,1])$ which is invariant under both M and T. If there is some $f_0 \in F$ with $\int_0^1 f_0(t)\,dt \neq 0$, then $T(f_0)$ is a non-zero multiple of g and since $T(f_0) \in F$, we have $g \in F$. But then for $n = 1, 2, \ldots, M^n(g) = g^{n+1}$ are all in F so that $F = L^2([0,1])$. On the other hand, if $\int_0^1 f(t)\,dt = 0$ for all $f \in F$, then for every $f \in F$, $M^n(f) = g^n f \in F$ so that $\int_0^1 t^n f(t)\,dt = 0$ for all $n = 0, 1, 2, \ldots$. Hence $f = 0$ and we see that $F = \{0\}$.

Now since T is itself compact, the earlier result applies with $m=2$, $S_1 = M$ and $S_2 = T$. Thus, if convergence holds on M, T and $M^2 + T^* \circ T$, where

$$(M^2 + T^* \circ T)(f)(x) = x^2 f(x) + \frac{x}{2}\int_0^1 (tf(t) + \tfrac{1}{6}f(t))\,dt,$$

then it holds on all compact operators on $L^2([0,1])$.

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